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LETTER TO THE EDITOR

Covariant differential calculus on the quantum superplane and generalization of the Wess–Zumino formalism

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Abstract. It is shown that the construction of covariant differential calculus on quantum planes developed by Wess and Zumino for arbitrary quadratic algebras can generalize to the quantum superplane if one accounts for the differences due to the graduation of the superplane.

In this work we discuss a differential calculus on the quantum superplane [1] which is covariant with respect to the action of a multiparametric quantum deformation of a general linear supergroup. We follow the construction recently published in a remarkable paper by Wess and Zumino [2].

In this paper [2] a differential calculus is developed for arbitrary quadratic algebras of which, as we show here, the quantum superplane can be considered as a special example. What is required, of course, is to take into account the differences due to the graduation of the superplane. In this way the central statement of our work is that the differential calculus on the quantum superplane leads to its dual in the sense of Manin [1] and is a superspace generalization of the profound observation, made by Wess and Zumino, that by establishing a differential calculus on the quantum (hyper-) plane the differentials of the basic variables can be identified with the dual quadratic algebra of Kobyzev and Manin [3].

Consider an associative algebra whose supernumerary elements x^I ($\hat{I}=0$ for $I=1,2,\ldots,n$ and $\hat{I}=1$ for $I=n+1,\ n+2,\ldots,n+m$) satisfy the quadratic relations (it is understood in the following that, unless specified to the contrary, repeated indices are summed over all the variable)

$$x^Ix^J-(-1)^{\hat{I}\hat{J}}B^{IJ}_{KL}x^Kx^L=0 \qquad I,J=1,2,\ldots,n+m. \tag{1}$$

A definition of the exterior derivative operation d satisfying $d^2=0$ and the graded Leibniz rule is required.

In general the differentials $X^I=\mathrm{d}x^I$ will commute neither with xs nor with functions $f(x)$ (formally defined as ordered power series) of the variables. In order to define the calculus let us write

$$F(x)X^J=(-1)^{\hat{F}(\hat{J}+1)}X^IO^J_IF(x) \qquad J=1,2,\ldots,n+m. \tag{2}$$

The linear differential operators O^J_I introduced in (2) must obey the 'linear' and 'quadratic' consistency conditions on the calculus:

$$(\delta^I_K\delta^J_L-(-1)^{\hat{I}\hat{J}}B^{IJ}_{KL})(\delta^K_Mx^L+(-1)^{\hat{K}\hat{L}}O^L_Mx^K)=0 \tag{3}$$

$$(-1)^{\hat{K}(\hat{P}+\hat{N})}(\delta^I_K\delta^J_L-(-1)^{\hat{I}\hat{J}}B^{IJ}_{KL})(O^P_Mx^K)(O^N_Px^L)=0$$

$$I,J,M,N=1,2,\ldots,n+m. \tag{4}$$

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They must be true if the relations (2) are to allow us to pull d/Xs through from one side to the other of the quadratic relations (1) (without generating any terms linear in the differentials). These resulting weak equalities denoted $\approx$ are so called because they hold at least by virtue of the basic relations.

On the superplane, (3) and (4) are replaced by strong conditions because Ox is a linear combination of the plane variables.

$$O_I^K x^J = C_{IL}^{JK} x^L \qquad I, J, K = 1, 2, \ldots, n+m \tag{5}$$

C being a matrix of purely numerical coefficients. The linear condition translates into

$$(\delta_K^I \delta_L^J - (-1)^{\hat{I}\hat{J}} B_{KL}^{IJ} (\delta_M^K \delta_N^L + (-1)^{\hat{K}\hat{L}} C_{MN}^{KL}) = 0$$
$$I, J, M, N = 1, 2, \ldots, n+m \tag{6}$$

whereas the corresponding strong relation which replaces the quadratic condition has the structure of a quantum Yang–Baxter equation (YBE). Under certain technical assumptions on B and C, the latter assumes a neat form (in the standard tensor notation)

$$\hat{B}_{23} \hat{C}_{13} \hat{C}_{12} = \hat{C}_{12} \hat{C}_{13} \hat{B}_{23} \tag{7}$$

in terms of hatted matrices

$$\hat{B} = PB \qquad \text{or} \qquad \hat{B}_{KL}^{IJ} = (-1)^{\hat{I}\hat{J}} B_{KL}^{JI} \qquad (\text{idem } C) \tag{8}$$

where P is the permutation matrix.

In arriving at (7) we assume that in any two-dimensional subspace signified by (I, J) the only non-vanishing elements of B (similarly, of C) are

$$B_{II}^{II} \cdot B_{JJ}^{JJ} \cdot B_{IJ}^{IJ} \cdot B_{JI}^{IJ} \cdot B_{IJ}^{JI} \cdot B_{JI}^{JI}.$$

This assumption is compatible with what is assumed for the R matrix (which is linearly related to B and C) in trying to solve the YBE to give in $n+m$ dimensions the R matrix in the following form (no summation over repeated indices)

$$R_{KL}^{IJ} = R_{IJ}^{IJ} \delta_K^I \delta_L^J + (1 - 1/r) \Theta(I-J) \delta_L^I \delta_K^J \tag{9}$$

with

$$R_{IJ}^{IJ} = \begin{cases} (-1)^{\hat{I}\hat{J}}/q_{IJ} & I < J \\ 1 & I = J = 1, 2, \ldots, n \\ -1/r & I = J = n+1, n+2, \ldots, n+m \\ (-1)^{\hat{I}\hat{J}} q_{JI/r} & I > J. \end{cases}$$

(For assumptions of a similar nature leading to the solution of the YBE in two dimensions see Schirrmacher *et al* [4].) According to our notation $\Theta(I-J) = 1$ if $I > J$ and zero if $I \leqslant J$. Equation (9) gives the multiparametric R matrix for the general linear quantum supergroup $\mathrm{GL}_{q_{IJ};r}(n, m)$. The $\binom{n+m}{2} + 1$ quantum deformation parameters are denoted

$$q_{IJ} \ (I < J) \qquad \text{and} \qquad r \ (\neq -1) \qquad I, J = 1, 2, \ldots, n+m.$$

Let us discuss the differential calculus which is the outcome of the R matrix (9). The quadratic consistency condition (7) on the calculus emerges as a consequence of the YBE,

$$R_{23} R_{13} R_{12} = R_{12} R_{13} R_{23}. \tag{10}$$

This condition admits two solutions. We consider the following solution

$$B = \hat{R} \tag{11}$$

$$C = r\hat{R}. \tag{12}$$

The linear consistency requirement (6) is also fulfilled because it is easy to verify that the R matrix satisfies

$$(\delta^I_K\delta^J_L-(-1)^{\hat I\hat J}\hat R^{IJ}_{KL})(\delta^K_M\delta^L_N+(-1)^{\hat K\hat L}r\hat R^{KL}_{MN})=0$$
$$I, J, M, N=1,2,\ldots,n+m. \tag{13}$$

Following Wess and Zumino [2], the various types of deformed (anti-) commutation relations in the calculus are expressible in terms of the R matrix (9). In addition to the 'intermediary' relations

$$x^IX^J=(-1)^{\hat I(\hat J+1)}r\hat R^{IJ}_{KL}X^Kx^L \tag{14}$$

we have the basic relations

$$x^Jx^J=(-1)^{\hat I\hat J}\hat R^{IJ}_{KL}x^Kx^L \tag{15}$$

and their dual

$$X^IX^J=(-1)^{I(\hat J+1)+1+\hat K}r\hat R^{IJ}_{KL}X^KX^L \qquad I, J=1,2,\ldots,n+m. \tag{16}$$

In addition if one introduces derivatives of the superplane ∂s through $d=X^K\partial_K$, then one can complete the scheme by giving also the deformed (anti-) commutation relations between derivatives, between derivatives and variables and between derivatives and differentials. They are (following considerations analogous to [2])

$$\partial_I\partial_J=(-1)^{\hat I\hat J}\hat R^{LK}_{JI}\partial_K\partial_L \tag{17}$$

$$\partial_Jx^I=\delta^I_J+(-1)^{\hat I\hat K}r\hat R^{IK}_{JL}x^L\partial_K \tag{18}$$

$$\partial_J=X^I=(-1)^{\hat J(\hat I+1)}(\hat R^{IK}_{JL}+(-1)^{\hat I\hat K}(1/r-1)\delta^I_J\delta^K_L)X^L\partial_K. \tag{19}$$

To obtain the explicit form of the above relations we can insert the R matrix (9) into (14)–(19). It is easy to check that the basic variables span a q_{IJ} superplane and their differentials span a $1/p_{IJ}$ exterior superplane, i.e (no summation) $I<J$

$$x^Ix^J=(-1)^{\hat I\hat J}q_{IJ}x^Jx^I \tag{20}$$

$$X^IX^J=(-1)^{(\hat I+1)(\hat J+1)}1/p_{IJ}X^JX^I \tag{21}$$

$$(X^1)^2=(X^2)^2=\ldots=(X^n)^2=(x^{n+1})^2$$
$$=(x^{n+2})^2=\ldots=(x^{n+m})^2=0.$$

Equations (20) and (21) are identical in structure to the corresponding defining quadratic relations for the quantum superspace and its dual given by Manin [1] (we have derived these relations independently in the context of differential calculus). As compared with [1] we have an extra degree of freedom (in the choice of r) which we can exercise to choose p_{IJ} maintaining

$$p_{IJ}q_{IJ}=r. \tag{22}$$

If $r=1$, p_{IJ} is inversely related to q_{IJ}. Other interesting choices are $p_{IJ}=q_{IJ}$ or $p_{IJ}=1$. This requires that $q_{IJ}=q$ independently of the values of I and J.

The calculus satisfying relations (14)–(19) is invariant under the action of $\mathrm{GL}_{q_{IJ};r}(n,m)$ viewed as endomorphisms of the superplane. We have checked that the resulting quantized group structure induced by the R matrix (9) on the fundamental matrix space includes as special cases the (anti-) commutation structures of Manin [1], of Schwenk *et al* [5], and of Corrigan *et al* [6].

In this work we have considered the real superplane, i.e. $\overline{x^I} = x^I$. It is easy to check that the entire scheme of relations (14)–(19) and their adjoint system coincide if

$$\begin{aligned}
&\bar{q}_{IJ} = 1/q_{IJ} \qquad \bar{p}_{IJ} = 1/p_{IJ} \\
&\overline{x^I} = x^I \qquad \overline{X^I} = (-1)^{\hat{I}} X^I \qquad \bar{d} = -d \\
&\bar{\partial}_I = \begin{cases} -r^{(n-m-I+1)}\partial_I & I = 1, 2, \ldots, n \\ +r^{(I-n-m)}\partial_I & I = n+1, n+2, \ldots, n+m. \end{cases}
\end{aligned} \tag{23}$$

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